PC2 corrected

Question #1

Under perfect competition, profit optimization is given by equalizing factor prices with their marginal productivity:

$$\frac{w}{p_1} = \frac{\partial Y_1}{\partial L_1} = (1-\alpha)k_1^{\alpha} \;;\; \frac{w}{p_2} = \frac{\partial Y_2}{\partial L_2} = (1-\beta)k_2^{\beta}$$

$$\frac{r}{p_1} = \frac{\partial Y_1}{\partial K_1} = \alpha k_1^{\alpha-1} \;;\; \frac{r}{p_2} = \frac{\partial Y_2}{\partial K_2} = \beta k_2^{\beta-1}$$

This implies:

$$\omega = \frac{w}{r} = \frac{1-\alpha}{\alpha}k_1 = \frac{1-\beta}{\beta}k_2$$

Or otherwise:

$$\boxed{k_1 = \frac{\alpha}{1-\alpha}\omega; \quad k_2 = \frac{\beta}{1-\beta}\omega}$$

For a given relative price of labor ω, the capital-labor ratio is higher in sector 2 than in sector 1 because $\beta > \alpha$: sector 2 is capital intensive (for a given quantity of L and K, the marginal productivity of capital is higher in sector 2, leading to a higher capital-labor ratio at the producer's equilibrium).

An increase in the relative cost of labor (*i.e.* a rise in ω) leads the firm to substitute capital for labor. This is more the case in sector 2 than in sector 1 because sector 2 is less labor intensive: less units of capital are necessary to substitute one unit of labor in sector 2 than in sector 1, so there is more incentive to substitute.

At the producer's equilibrium, we thus have:

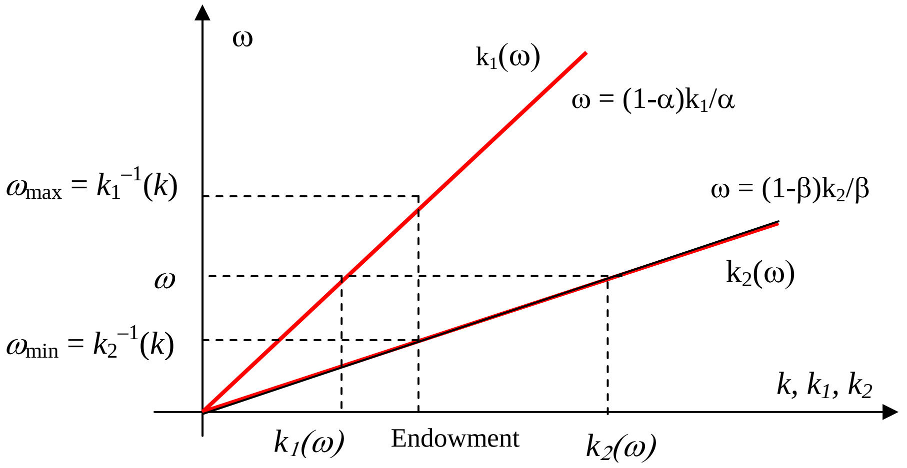
$$\frac{k_2}{k_1} = \frac{\beta}{1-\beta}\frac{1-\alpha}{\alpha} = \kappa > 1$$

For a given ω (given by the market) the optimal capital/labor ratio is higher in sector 2. Again, this is because sector 2 is more capital intensive: for a given quantity of capital, the marginal productivity is higher than in sector 1, hence the firm chooses to use more capital.

$k_i(\omega)$ is the optimal capital/labor mix for a given labor/capital compensation ratio. It increases with ω, all the more so in sector 2 than sector 1. As a result, $k_1^{-1}(\omega)$ is steeper than $k_2^{-1}(\omega)$. Specialization depends on ω, with two corner solutions:

- For $\omega_{\min} = \frac{1-\beta}{\beta}k$, the optimal capital/labor ratio of sector 2 coincides with the capital/labor endowment of the country. In this case, the country specializes fully into good 2. For $\omega < \omega_{\min}$, the country also specializes into good 2 since the relative price of labor is closer to $k_2^{-1}(k)$ then to $k_1^{-1}(k)$.

- For $\omega_{\max} = \dfrac{1-\alpha}{\alpha} k$, the optimal capital/labor ratio of sector 1 coincides with the capital/labor endowment of the country. In this case, the country specializes fully into good 1. For $\omega > \omega_{\max}$, the country also specializes into good 1 since the relative price of labor is closer to $k_1^{-1}(k)$ than to $k_2^{-1}(k)$.
- For $\omega_{\min} < \omega < \omega_{\max}$, the optimal capital/labor ratio of sector 1 is lower than the capital/labor endowment of the country, whereas the optimal capital/labor ratio of sector 2 is higher than the capital/labor endowment. In this case, the country produces the two goods in quantities such as the weighted capital/labor ratio coincides with the endowment.



Question #2

In perfect competition, the representative firm considers the relative price of factors ω as given. It follows that k_1 and k_2 are given (they depend on ω). One can compute the MTR by differentiating production functions $Y_1 = k_1^{\alpha} L_1$ and $Y_2 = k_2^{\beta} L_2$:

$$MTR = -\frac{\partial Y_2}{\partial Y_1} = -\frac{\partial Y_2}{\partial L_2}\frac{\partial L_2}{\partial L_1} / \frac{\partial Y_1}{\partial L_1} = \frac{k_2^{\beta}}{k_1^{\alpha}} = \left(\frac{\beta}{1-\beta}\right)^{\beta}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\omega^{\beta-\alpha}$$

The MRT is an increasing function of w: the higher the relative price of labor, the larger the fall in Y_2 involved by a marginal rise in Y_1. This is because a rise in ω leads to k_2 increasing more than k_1: there is more capital substitution in sector 2 than in sector 1. Then, the marginal productivity of labor is higher in sector 2 for a higher level of ω: reducing employment in sector 2 is more painful for production in this sector than the same reduction of employment when ω is low.

In the competitive equilibrium, $MTR = p$, thus: $\dfrac{dp}{p} = (\beta-\alpha)\dfrac{d\omega}{\omega}$.

Since $\beta > \alpha$, a rise in p (relative price of good 1) triggers a rise in the relative compensation of labor, which is used more intensively in sector 1. This is the **Stolper-Samuelson theorem**.

Note that the MTR is the slope of the PPF, i.e. the locus of all possible production combinations. Along the PPF, we have $k_1 = \alpha/(1-\alpha)\omega$ and $k_2 = \beta/(1-\beta)\omega$.

Question #3

We use the following relationships: $K_1 = k_1 L_1$; $K_2 = k_2 L_2$; $K = K_1 + K_2$; and $L = L_1 + L_2$. This implies: $K_2 = k_2 L_2 = K - K_1 = K - k_1 L_1 = K - k_1 (L - L_2)$

Similarly: $K_1 = k_1 L_1 = K - K_2 = K - k_2 L_2 = K - k_2 (L - L_1)$

Dividing both identities by L, we get:

$$\begin{cases} k_1 \dfrac{L_1}{L} = k - k_2 \left(1 - \dfrac{L_1}{L}\right) \\ k_2 \dfrac{L_2}{L} = k - k_1 \left(1 - \dfrac{L_2}{L}\right) \end{cases}$$

Hence: $\begin{cases} (k_1 - k_2) \dfrac{L_1}{L} = k - k_2 \\ (k_2 - k_1) \dfrac{L_2}{L} = k - k_1 \end{cases}$

We get: $\begin{cases} \dfrac{L_1}{L} = \dfrac{k - k_2}{k_1 - k_2} \\ \dfrac{L_2}{L} = \dfrac{k - k_1}{k_2 - k_1} \end{cases}$ with $k_1 < k < k_2$

If ω is constant, then k_1 and k_2 are constant too. The capital/labor ratio resulting from factor endowments (k) is also constant. Hence it is easy to differentiate:

$$\begin{cases} L_1 = \left(\dfrac{k - k_2}{k_1 - k_2}\right) L = \dfrac{K}{k_1 - k_2} - \dfrac{k_2}{k_1 - k_2} L \\ L_2 = \left(\dfrac{k - k_1}{k_2 - k_1}\right) L = \dfrac{K}{k_2 - k_1} - \dfrac{k_1}{k_2 - k_1} L \end{cases}$$

We get: $\begin{cases} dL_1 = -\dfrac{k_2}{k_1 - k_2} dL \\ dL_2 = -\dfrac{k_1}{k_2 - k_1} dL \end{cases}$

We can already see that a rise in L triggers a rise in L_1 but a fall in L_2.

Let us now come back to the production function:

$Y_1 = k_1^{\alpha} L_1$ and $Y_2 = k_2^{\beta} L_2$

And differentiate it:

$$\begin{cases} dY_1 = k_1^{\alpha} dL_1 = \dfrac{k_1^{\alpha} k_2}{k_2 - k_1} dL \\ dY_2 = k_2^{\alpha} dL_2 = -\dfrac{k_2^{\beta} k_1}{k_2 - k_1} dL \end{cases}$$

When the workforce increases exogenously, production increases in the relatively labor-intensive industry and decreases in the relatively capital-intensive industry. This is the **Rybczynski theorem**.

Question #4

$$\frac{Y_1}{Y_2} = \frac{k_1^{\alpha} L_1}{k_2^{\beta} L_2} = \frac{k_1^{\alpha}\left(-\frac{k-k_2}{k_2-k_1}L\right)}{k_2^{\beta}\left(\frac{k-k_1}{k_2-k_1}L\right)} = -\frac{k_1^{\alpha}}{k_2^{\beta}}\left(\frac{k-k_2}{k-k_1}\right) \qquad \text{and} \qquad \frac{pC_1}{C_2} = 1$$

Hence in autarky: $\frac{Y_1}{Y_2} = \frac{C_1}{C_2} = \frac{1}{p} = -\frac{k_1^{\alpha}}{k_2^{\beta}}\left(\frac{k-k_2}{k-k_1}\right)$

And: $p = -\frac{k_2^{\beta}}{k_1^{\alpha}}\left(\frac{k-k_1}{k-k_2}\right) = -\left(\frac{\beta}{1-\beta}\right)^{\beta}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\omega^{\beta-\alpha}\left(\frac{k-\frac{\alpha}{1-\alpha}\omega}{k-\frac{\beta}{1-\beta}\omega}\right)$

Now we have two relations between p and ω (this one and the one obtained from question 2). We can derive ω and p in autarky and compare p with the relative price of good 1 on the international market.

Question #5

As a consequence of trade, the relative price of good 1 goes down in the small economy, which reduces the relative compensation of labor. This is the **Stolper-Samuelson theorem**.